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A q-analogue of the embedding chain $U(6) \supset G \supset SO(3)$

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Abstract

A q-analogue of the embedding chains of the Arima–Iachello model is proposed. The generators of the deformed $U(6)$ subalgebras are written in terms of the generators of $gl_q(6)$, using q-bosons.

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1. Introduction

Since their introduction [1, 2], the quantum algebras G_q or $U_q(G)$, i.e. the q-deformed universal enveloping algebras of a semi-simple Lie algebra G have been a topic of active research both in physics and mathematics. The underlying idea in some of their applications is to use a q-deformed algebra instead of a Lie algebra to realize a *generalized dynamical symmetry*. For a review of the applications and methods of the dynamical or spectrum generating algebras in physics, see [3], and in nuclear physics [4]. The key idea of a dynamical symmetry scheme is to write the Hamiltonian of a physical system as a sum of invariants, usually second order Casimir C, with constants to be determined by experimental data, of the embedding chains of algebras of the type:

$$G \supset L \supset \cdots \supset SO(3) \tag{1}$$

$$\mathcal{H} = C(G) + C(L) + \cdots + C(SO(3)) \tag{2}$$

where $SO(3)$ describes the angular momentum and, usually, the Casimir operators are written using Jordan–Schwinger-like realization of the algebra G by means of bosonic creation–annihilation operators. The idea of dynamical symmetry has countless applications in

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molecular, atomic, nuclear, hadronic and chemical physics. The most simple example is the rigid rotator where the Hamiltonian is written as the Casimir operator of $SU(2)$

$$C = kJ(J+1). \tag{3}$$

The energy spectrum of equation (3) is of the form

$$E_j = kj(j+1). \tag{4}$$

The replacement in equation (3) of the Casimir of $SU(2)$ by the Casimir operator of $sl_q(2)$ [5] provides the first example of application of deformed algebra as dynamical symmetry (see below for notation)

$$C = K[J]_q[J+1]_q. \tag{5}$$

Now the energy spectrum will depend on a parameter $q = \exp \mathrm{i}\tau$ and we have

$$E_j^q = K[j]_q[(j+1)]_q = K\frac{\sin(|\tau|j)\sin(|\tau|(j+1))}{\sin^2(|\tau|)}. \tag{6}$$

Equation (6) fits experimental data for several deformed nuclei [5], better than equation (4) ($|\tau| = 0$). The results of this simple model suggest that it may be worthwhile to further investigate the idea of generalized dynamical symmetry based on q-algebras. Indeed in this last decade many applications, mainly, but not uniquely, in molecular and nuclear physics have been investigated. For an excellent review of the subject with an exhaustive list of references see [6]. The simplest, non-trivial, embedding chain is the so-called Elliot model:

$$SU(3) \supset SO(3). \tag{7}$$

The deformation of the simple embedding chain of equation (7) is not at all an easy task. Indeed in [7], it has been shown that the generators of $so_q(3)$ can be expressed by means of the generators of $gl_q(3)$, not of $sl_q(3)$, iff the algebraic relations are restricted to the symmetric representations. Moreover the coproduct, which is essential to define the tensor product of spaces, of $gl_q(3)$ *does not induce* the standard coproduct on $so_q(3)$. The q-analogue of the chain equation (7) has been widely studied. In [8] the author has proposed a possible solution, but the problem has been tackled from several points of view, see [6] for references to the different solutions and for physical applications of the q-analogue of the embedding chain equation (7). Therefore, it is clear that an essential step to carry forward the programme of application of q-algebras as *generalized dynamical symmetry*, beyond the simple models above discussed, is to have at our disposal a formalism which allows us to build up analogous chains to equation (1) replacing the Lie algebras by the deformed ones. Of course, as we are no longer dealing with Lie algebras, the term *embedding* has to be intended in the loose sense that the generators of the embedded deformed subalgebra are expressed in terms of the generators of the algebra while the Hopf structure can be inherited from that of the embedding algebra or imposed on the generators of the embedded algebra. The root of the problem, as has been discussed in [9], lies in the fact that G_q are well defined only in the Cartan–Chevalley basis and this basis is not suitable to discuss embedding of subalgebras except the regular ones. The classification of so-called singular subalgebras of Lie algebras has been started by Dynkin and we refer to the clear paper of Gruber and Lorente [10], where the embedding matrices are explicitly computed for low rank algebras. In particular, in [9] it has been shown that, in the case where the rank of L, maximal singular algebra of G, is equal to the rank of G minus one, it is possible, using realization of G_q in terms of q-bosons and/or in terms of the so-called q-fermions, to write the Cartan–Chevalley generators of L_q in terms of the generators of G_q. Let us remark that this result is *a priori* not at all obvious due to the non-linear structure of G_q. It has also been discussed what kind of deformed G is obtained if the standard coproduct is imposed on the generators of L_q in the standard way instead of being derived from that of G_q. The aim of this paper is to focus on the embedding chains appearing in the interacting boson (IBM) or Arima–Iachello model [11, 12] and to discuss in which sense one can write analogous chains of q-algebras. This very successful model is based on the following three embedding chains

$$SU(6) \supset \begin{cases} SU(5) \supset SO(5) \supset SO(3) & \text{(vibrational)} \\ SU(3) \supset SO(3) & \text{(rotational)} \\ SO(6) \supset SO(5) \supset SO(3) & (\gamma\text{-unstable}). \end{cases} \tag{8}$$

A partial answer to the question of finding a q-analogue of the above embedding chains has been given in [13], where it has been shown that, using q-bosons realization, the deformed maximal $SU(6)$ subalgebras, i.e. $sl_q(3)$, $so_q(6)$, can be written in terms of the generators of $gl_q(6)$ and that this procedure can be extended also to the deformation of $SO(5)$, maximal subalgebra of $SU(5) \subset SU(6)$. It should be mentioned that deformed versions of the Arima–Iachello model have already appeared in the literature. To this aim in [14], use of the notion of complementary subalgebras introduced about 30 years ago by Moshinsky and Quesne [17] has been made. Two subalgebras L_1 and L_2 of an algebra G are complementary in one definite irrep of G, if there is a one-to-one correspondence between the irreps of L_1 and L_2 contained in the considered irrep of G. Using this notion in [14] a Hamiltonian has been written in terms of the second order Casimir of $su_q^{sd}(1,1)$, $su_q^{d}(1,1)$ and $su_q(2)$, where s, d are boson operators and the second $su(1,1)$ is contained in the first one. This Hamiltonian, in the limit $q \to 1$, for a particular value of the coefficients of the Casimir operators, tends to a Hamiltonian in the γ-unstable chain. In [15], it has been shown that this model represents the general IBM Hamiltonian, the q-deformation parameter behaving as a symmetry mixing parameter. In [16], using the notion of complementary subalgebras and of deforming functionals, see below, q-deformations of the Hamiltonian of the IBM model in the three limits have been written. The Hamiltonian depends on three deformation parameters, which are fitted by experimental data. As an example, in [16] fits of the energy spectra (with the Hamiltonian depending on two deformation parameters) and of the $E2$-transition rates and comparisons with experimental data for nuclei $^{110-114}$Cd, $^{190-196}$Pt and $^{124-128}$Xe have been carried out. Although these approaches are interesting, it should be pointed out that the content and the embedding of subalgebras is not well defined. Rather than a deformation of the embedding chains of equation (8), they look like an interesting dynamical model based on q-algebras and inspired by the IBM model. Our aim is to build up the whole algebraic construction of a q-analogue of the chains of equation (8). From the construction it should be possible to write Hamiltonians which in the limit $q \to 1$ tend to Hamiltonians of the undeformed model, for any value of the coefficients of the Casimir operators. Indeed, we shall show that, replacing $U(6)$ by $gl_q(6)$, we can write the generators of any deformed $U(6)$ subalgebra in terms of the generators of $gl_q(6)$. To make the paper self-contained, in section 2 we briefly recall the formulae we shall use in the following. In section 3 we write explicitly the q-analogue of the embedding chains equation (8). In section 4 we summarize and discuss our results.

2. Reminder

2.1. Deformed algebras

Let us recall, also to fix the notation, the definition of G_q associated with a simple Lie algebra G of rank r defined by the Cartan matrix (a_{ij}) in the Chevalley basis. G_q is generated by $3r$ elements $e_i^\pm$ and h_i which satisfy $(i, j = 1 = 1, \ldots, r)$

$$[e_i^+, e_j^-] = \delta_{ij}[h_i]_{q_i} \qquad [h_i, h_j] = 0 \qquad [h_i, e_j^\pm] = \pm a_{ij} e_j^\pm \tag{9}$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{10}$$

and $q_i = q^{d_i}$, d_i being non-zero integers with greatest common divisor equal to 1 such that $d_i a_{ij} = d_j a_{ji}$. For simple laced algebras $d_i = 1$ while for $so_q(2n+1)$ $(sp_q(2n))$ $d_i = 2\,(1)$, $i \neq n$, $d_n = 1\,(2)$. Further the generators have to satisfy the Serre relations:

$$\sum_{0\leqslant n\leqslant 1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} (e_i^{\pm})^{1-a_{ij}-n} e_j^{\pm} (e_i^{\pm})^n = 0 \tag{11}$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q![n]_q} \qquad [n]_q! = [1]_q\,[2]_q \ldots [n]_q. \tag{12}$$

Let us recall the definition of $gl(n)_q$ $(i, j = 1, 2, \ldots, n-1, k = 1, 2, \ldots, n-1, n)$:

$$\begin{aligned} &[e_i^+, e_j^-] = \delta_{ij}[n_i - n_{i+1}]_q \qquad [n_i, n_k] = 0 \\ &[n_k, e_j^{\pm}] = \pm(\delta_{k,j} - \delta_{k-1,j}). \end{aligned} \tag{13}$$

The Serre relations are computed using $a_{k,j} = -(\delta_{k-1,j} + \delta_{k,j-1})$. In the following we assume $h_i = (h_i)^\dagger$ and the deformation parameter q to be different from the roots of the unity. The algebra G_q is endowed with a Hopf algebra structure, i.e. on G_q the action of the coproduct Δ, antipode S and co-unit ε is defined. This extremely relevant aspect will not be discussed here. We recall only the definition of the coproduct which we shall briefly refer to in the following:

$$\Delta(h_i) = h_i \otimes \mathbf{1} + \mathbf{1} \otimes h_i \qquad \Delta(e_i^{\pm}) = e_i^{\pm} \otimes q_i^{h_i/2} + q_i^{-h_i/2} \otimes e_i^{\pm}. \tag{14}$$

2.2. *q-bosons*

Let us recall the definition of Biedenharn–MacFarlane q-bosons [18, 19] which we denote by b_i^+, b_i

$$b_i b_j^+ - q^{\delta_{ij}} b_j^+ b_i = \delta_{ij} q^{-N_i} \tag{15}$$

$$[N_i, b_j^+] = \delta_{ij} b_j^+ \qquad [N_i, b_j] = -\delta_{ij} b_j \qquad [N_i, N_j] = 0. \tag{16}$$

The explicit construction of q-bosons in terms of non-deformed standard bosonic oscillators $(\tilde{b}_i^+, \tilde{b}_i)$ is [20]

$$b_i^+ = \sqrt{\frac{[N_i]_q}{N_i}}\tilde{b}_i^+ \qquad b_i = \tilde{b}_i \sqrt{\frac{[N_i]_q}{N_i}}. \tag{17}$$

Remark that, if $\tilde{b}_i^+$ is the adjoint of $\tilde{b}_i$, then b_i^+ is the adjoint of b_i iff q is real or $q = \exp \mathrm{i}\tau$, τ real.

2.3. *q-boson realizations of $sl_q(2)$ and $so_q(3)$*

In order to clarify what we mean by $sl_q(2)$ and $so_q(3)$, let us write explicitly the q-boson realization of $sl_q(2)$

$$J_+ = b_1^+ b_2 \qquad J_- = b_2^+ b_1 \qquad 2J_0 = N_1 - N_2 \tag{18}$$

the states of the irreducible representation (j, m) in the corresponding Fock space are

$$\psi_{jm} = \frac{(b_1^+)^{j+m}(b_2^+)^{j-m}}{\sqrt{[j+m]_q![j-m]_q!}}\psi_0 \tag{19}$$

and of $so_q(3)$

$$\begin{aligned} L_+ &= q^{N_{-1}}q^{-N_0/2}\sqrt{q^{N_1}+q^{-N_1}}b_1^+b_0 + b_0^+b_{-1}q^{N_1}q^{-N_0/2}\sqrt{q^{N_{-1}}+q^{-N_{-1}}} \\ L_- &= b_0^+b_1q^{N_{-1}}q^{-N_0/2}\sqrt{q^{N_1}+q^{-N_1}} + q^{N_1}q^{-N_0/2}\sqrt{q^{N_{-1}}+q^{-N_{-1}}}b_{-1}^+b_0 \\ L_0 &= N_1 - N_{-1} \end{aligned} \tag{20}$$

the states of the odd-dimensional irreducible representation $(L, m = n_1 - n_{-1})$ $(L = \max\{n_1\} = n_1 + n_0 + n_{-1})$ in the corresponding Fock space are linear combinations of

$$\psi_{n_1,n_0,n_{-1}} = \frac{(b_1^+)^{n_1}(b_0^+)^{n_0}(b_{-1}^+)^{n_{-1}}}{\sqrt{[n_1]_q![n_0]_q![n_{-1}]_q!}}\psi_0. \tag{21}$$

2.4. *q-tensor operator*

From the formula of coproduct equation (14) Biedenharm and Tarlini [22], see also [23] and [24], have derived the general structure of q-tensor operators for $sl_q(2)$,

$$\begin{aligned} \left[J_\pm, T_m^k(q)\right]_{q^{-m}} q^{-J_0} &= \sqrt{[k \mp m]_q[k \pm m+1]_q}T_{m\pm1}^k(q) \\ \left[J_0, T_m^k(q)\right] &= mT_m^k(q) \end{aligned} \tag{22}$$

where the q-commutator is defined as

$$[A, B]_q = AB - qBA. \tag{23}$$

2.5. *Deforming map between sl(2) and $sl_q(2)$*

In [25], an invertible deforming functional $\mathcal{Q}_\pm$ has been introduced which allows us to relate $sl(2) \Leftrightarrow sl_q(2)$. Denoting by a lower case (resp. capital) letter $j_{\pm,0}$ $(J_{\pm,0})$ the generator of $sl(2)$ $(sl_q(2))$ it is possible to write (q real)

$$J_+ = \mathcal{Q}_+(j_\pm, j_0)j_+ \qquad J_- = (J_+)^\dagger \qquad J_0 = j_0 \tag{24}$$

where

$$\mathcal{Q}_+ = \sqrt{\frac{\{[J_0+\mathbf{J}]_q[J_0-\mathbf{J}-1]_q\}}{\{(j_0+\mathbf{j})(j_0-\mathbf{j}-1)\}}} \tag{25}$$

and the operator $\mathbf{j}(\mathbf{J})$ is defined by the Casimir operator of $sl(2)$ $(sl_q(2))$

$$C = \mathbf{j}(\mathbf{j}+1) \qquad (C_q = [\mathbf{J}]_q[\mathbf{J}+1]_q). \tag{26}$$

2.6. *Deforming map between SO(5) and $so_q(5)$*

In [26], in the space of the symmetric irreps, invertible deforming maps have been derived which allow us to express $sl_q(n)$ and $sp_q(2n)$, respectively, in terms of $U(n)$ and $Sp(2n)$. As the irreducible representations in the Fock space of bosons or q-bosons are symmetric and $SO(5) \equiv Sp(4)$, we report here explicitly the map $sp_q(4) \Leftrightarrow Sp(4)$ which we shall use in the following. Denoting by small (capital) letters the generators of deformed (undeformed) algebra, we have

$$
\begin{aligned}
e_1^+ &= E_1^+ \sqrt{\frac{[H_1 + H_2 + 1]_q [H_2]_q}{(H_1 + H_2 + 1) H_2}} \\
e_2^+ &= \frac{2}{q + q^{-1}} E_2^+ \sqrt{\frac{[(H_2 + 1]_q [-H_2 - 2)]_q}{(H_2 + 1)(-H_2 - 2)}}
\end{aligned}
\tag{27}
$$

$$
h_k = H_k \qquad k = 1, 2. \tag{28}
$$

3. *q*-embedding

In this section we discuss in detail the meaning of the q-embedding chain, L being a maximal singular subalgebra of G

$$
G_q \supset L_q \supset so_q(3) \tag{29}
$$

for the case where $G_q = gl_q(6)$. In the following we denote by a lower case (resp. capital) letters $e^{\pm}$, h ($E^{\pm}$, H) the generators of $gl_q(6)$ (L_q) and by $L_{\pm,0}$ the generators of $so_q(3)$. With a hat on $E^{\pm}$, H we denote, when required, the generators of a maximal subalgebra of L. Let us recall once again that equation (29) holds if the generators of L_q can be expressed in terms of the generators of G_q, at least in a particular realization of G_q, in the present work using a q-boson realization. In the following we assume q real, therefore any generator X^- is the adjoint of X^+ and the generators of the Cartan subalgebra are self-adjoint. We shall comment in the conclusions on the more general case.

Let us recall the q-boson realization of $sl_q(6) \subset gl_q(6)$ ($i = 1$–5)

$$
e_i^+ = b_i^+ b_{i+1} \qquad h_i = N_i - N_{i+1}. \tag{30}
$$

To get $gl_q(6)$ one has to add to the previous generators

$$
h_0 = \sum_{i=1}^{5} h_i = \sum_{j=1}^{6} N_j. \tag{31}
$$

In the following we write explicitly the q-analogue of the embedding chains of the Arima–Iachello mode. The notation is self-explanatory. We use a vertical arrow in the equations to point out what the embedding chains tend to in the limit $q \to 1$. In the following, to save space in some equations we shall denote by e_i^+ the generators of $sl_q(6)$, whose content in q-bosons is given in equation (30). Let us recall also that the Cartan generators of the deformed and undeformed algebras are the same.

3.1. q-analogue of the vibrational embedding chain

$$
\begin{aligned}
&gl_q(6) \supset gl_q(5) \supset so_q(5) \\
&\qquad\qquad\qquad \supset^{\Uparrow_{q\to 1}} so_q(3)
\end{aligned}
\tag{32}
$$

Clearly, the generators of $gl_q(5)$ are obtained from those of $gl_q(6)$ neglecting $e_5^{\pm}$, h_5 and N_6 and the q-boson realization of $so_q(5) \subset gl_q(5) \subset gl_q(6)$ is [13]

$$
\begin{aligned}
E_1^\dagger &= \Big\{ \sqrt{q^{N_1}+q^{-N_1}}\, b_1^+ b_2 \sqrt{q^{N_2}+q^{-N_2}}\, q^{-(N_4-N_5)} \\
&\qquad + \sqrt{q^{N_4}+q^{-N_4}}\, b_4^+ b_5 \sqrt{q^{N_5}+q^{-N_5}}\, q^{(N_1-N_2)} \Big\} (q+q^{-1})^{-1} \\
E_2^\dagger &= q^{N_4-N_3/2} \sqrt{q^{N_2}+q^{-N_2}}\, b_2^+ b_3 + b_3^+ b_4 q^{N_2-N_3/2} \sqrt{q^{N_4}+q^{-N_4}} \\
H_1 &= N_1 - N_2 + N_4 - N_5 \\
H_2 &= 2(N_2 - N_4)
\end{aligned} \tag{33}
$$

the generators of $so_q(3)$ can be written in terms of the generators of $gl_q(5)$

$$
\begin{aligned}
L_+ &= \mathcal{Q}_+ \Bigg\{ 2 \left[\sqrt{\frac{N_1}{[N_1]_q}} e_1^+ \sqrt{\frac{N_2}{[N_2]_q}} + \sqrt{\frac{N_4}{[N_4]_q}} e_4^+ \sqrt{\frac{N_5}{[N_5]_q}} \right] \\
&\qquad + \sqrt{6} \left[\sqrt{\frac{N_2}{[N_2]_q}} e_2^+ \sqrt{\frac{N_3}{[N_3]_q}} + \sqrt{\frac{N_3}{[N_3]_q}} e_3^+ \sqrt{\frac{N_4}{[N_4]_q}} \right] \Bigg\} \\
L_0 &= 2N_1 + N_2 - N_4 - 2N_5 = 2H_1 + \tfrac{3}{2} H_2
\end{aligned} \tag{34}
$$

where H_1, H_2 are Cartan generators of $so_q(5)$ and we have used equations (17)–(30) and the deforming map equation (24).

3.2. *q-analogue of the rotational embedding chain*

$$
\begin{aligned}
gl_q(6) &\supset gl_q(3) \\
&\qquad \supset^{\Uparrow_{q\to 1}} so_q(3)
\end{aligned} \tag{35}
$$

the q-boson realization of $sl_q(3) \subset gl_q(6)$ [13],

$$
\begin{aligned}
E_1^\dagger &= \Big\{ q^{N_4-N_2/2} \sqrt{q^{N_1}+q^{-N_1}}\, b_1^+ b_2 + b_2^+ b_4 q^{N_1-N_2/2} \sqrt{q^{N_4}+q^{-N_4}} \Big\} q^{-(N_3-N_6)/2} + b_3^+ b_5 q^{(N_1-N_4)} \\
E_2^\dagger &= \Big\{ q^{N_6-N_5/2} \sqrt{q^{N_4}+q^{-N_4}}\, b_4^+ b_5 + b_5^+ b_6 q^{N_4-N_5/2} \sqrt{q^{N_6}+q^{-N_6}} \Big\} q^{(N_2-N_3)/2} + b_2^+ b_3 q^{-(N_4-N_6)} \\
H_1 &= 2N_1 - 2N_4 + N_3 - N_5 \\
H_2 &= N_2 - N_3 + 2N_4 - 2N_6
\end{aligned} \tag{36}
$$

the generators of $so_q(3)$ can be written in terms of the generators of $gl_q(6)$

$$
\begin{aligned}
L_+ &= \mathcal{Q}_+ \Bigg\{ 2 \left[\sqrt{\frac{N_1}{[N_1]_q}} e_1^+ \sqrt{\frac{N_2}{[N_2]_q}} + \sqrt{\frac{N_2}{[N_2]_q}} [e_2^+, e_3^+]_q \sqrt{\frac{N_4}{[N_4]_q}} q^{N_3} \right] \\
&\qquad + \sqrt{2} \left[\sqrt{\frac{N_3}{[N_3]_q}} [e_3^+, e_4^+]_q \sqrt{\frac{N_5}{[N_5]_q}} q^{N_4} + \sqrt{\frac{N_2}{[N_2]_q}} e_2^+ \sqrt{\frac{N_3}{[N_3]_q}} \right] \\
&\qquad + 2 \left[\sqrt{\frac{N_4}{[N_4]_q}} e_4^+ \sqrt{\frac{N_5}{[N_5]_q}} + \sqrt{\frac{N_5}{[N_5]_q}} e_5^+ \sqrt{\frac{N_6}{[N_6]_q}} \right] \Bigg\} \\
L_0 &= 2N_1 + N_2 - N_5 - 2N_6 = H_1 + H_2
\end{aligned} \tag{37}
$$

where H_1, H_2 are Cartan generators of $sl_q(3)$.

3.3. *q-analogue of the γ-unstable embedding chain*

$$
\begin{aligned}
gl_q(6) &\supset so_q(6) \\
&\qquad \supset^{\Uparrow_{q\to 1}} so_q(5) \\
&\qquad\qquad \supset^{\Uparrow_{q\to 1}} so_q(3)
\end{aligned} \tag{38}
$$

the q-boson realization of $so_q(6) \subset sl_q(6)$ [13]

$$
\begin{aligned}
E_1^\dagger &= b_2^+ b_4 q^{(N_3-N_5)/2} + b_3^+ b_5 q^{-(N_2-N_4)/2} \\
E_2^\dagger &= b_1^+ b_2 q^{(N_5-N_6)/2} + b_5^+ b_6 q^{-(N_1-N_2)/2} \\
E_3^\dagger &= b_2^+ b_3 q^{(N_4-N_5)/2} + b_4^+ b_5 q^{-(N_2-N_3)/2} \\
H_1 &= N_2 - N_4 + N_3 - N_5 \\
H_2 &= N_1 - N_2 + N_5 - N_6 \\
H_3 &= N_2 - N_3 + N_4 - N_5
\end{aligned}
\tag{39}
$$

the q-boson realization of $so_q(5)$, deformation of $SO(5)$ maximal subalgebra of $SO(6) \subset SU(6)$, using equation (27), can be written as

$$
\begin{aligned}
E_1^\dagger &= \left(\sqrt{\frac{N_1}{[N_1]_q}} e_1^+ \sqrt{\frac{N_2}{[N_2]_q}} + \sqrt{\frac{N_5}{[N_5]_q}} e_5^+ \sqrt{\frac{N_6}{[N_6]_q}}\right) \sqrt{\frac{[\hat{H}_1 + \hat{H}_2 + 1]_q [\hat{H}_2]_q}{(\hat{H}_1 + \hat{H}_2 + 1)\hat{H}_2}} \\
E_2^\dagger &= \left(\sqrt{\frac{N_2}{[N_2]_q}} [e_2^+, e_3^+]_q \sqrt{\frac{N_4}{[N_4]_q}} q^{N_3} + \sqrt{\frac{N_3}{[N_3]_q}} [e_3^+, e_4^+]_q \sqrt{\frac{N_5}{[N_5]_q}} q^{N_4} + \sqrt{\frac{N_2}{[N_2]_q}} e_2^+ \sqrt{\frac{N_3}{[N_3]_q}} \right. \\
&\quad \left. + \sqrt{\frac{N_4}{[N_4]_q}} e_4^+ \sqrt{\frac{N_5}{[N_5]_q}}\right) \frac{2}{q + q^{-1}} \sqrt{\frac{[\hat{H}_2 + 1]_q [-\hat{H}_2 - 2)]_q}{(\hat{H}_2 + 1)(-\hat{H}_2 - 2)}}
\end{aligned}
\tag{40}
$$

$$
\hat{H}_1 = N_1 + N_5 - N_2 - N_6 \qquad \hat{H}_2 = 2(N_2 - N_5) \tag{41}
$$

where $\hat{H}_1$, $\hat{H}_2$ are the Cartan generators of $so_q(5)$ and the generators of $so_q(3)$ can be written in terms of the generators of $gl_q(6)$

$$
\begin{aligned}
L_+ = \mathcal{Q}_+ \Bigg\{ & \sqrt{2} \left[\sqrt{\frac{N_1}{[N_1]_q}} e_1^+ \sqrt{\frac{N_2}{[N_2]_q}} + \sqrt{\frac{N_5}{[N_5]_q}} e_5^+ \sqrt{\frac{N_6}{[N_6]_q}}\right] \\
& + \sqrt{\frac{3}{2}} \left[\sqrt{\frac{N_2}{[N_2]_q}} [e_2^+, e_3^+]_q \sqrt{\frac{N_4}{[N_4]_q}} q^{N_3} + \sqrt{\frac{N_3}{[N_3]_q}} [e_3^+, e_4^+]_q \sqrt{\frac{N_5}{[N_5]_q}} q^{N_4} \right. \\
& \left. + \sqrt{\frac{N_2}{[N_2]_q}} e_2^+ \sqrt{\frac{N_3}{[N_3]_q}} + \sqrt{\frac{N_4}{[N_4]_q}} e_4^+ \sqrt{\frac{N_5}{[N_5]_q}}\right]\Bigg\}
\end{aligned}
\tag{42}
$$

$$
L_0 = 2N_1 + N_2 - N_5 - 2N_6 = \tfrac{3}{2}(H_1 + H_3) + 2H_2
$$

where H_1, H_2, H_3 are Cartan generators of $so_q(6)$. We recall that

$$
\hat{H}_1 = H_2 \qquad \hat{H}_2 = H_1 - H_3. \tag{43}
$$

4. Conclusions

Starting from a spectrum generating algebra

$$
U(6) \supset L \supset SO(3) \tag{44}
$$

we have shown that, using q-bosons, a deformed analogue of this chain can be obtained replacing $U(6)$ by $gl_q(6)$ and writing the generators of L_q ($L = SO(6), SO(5), SU(3), SO(5) \subset SO(6)$) and of $so_q(3)$ in terms of the generators of $gl_q(6)$ (or of $gl_q(5)$ for the vibrational chain) with coefficients taking values in $gl_q(6)$ (or $gl_q(5)$). In order to write our results we have used the relation between standard bosonic operators and q-bosons equation (17), the deforming map between $su(2)$ and $su_q(2)$ equation (24) and between $sp(4) \equiv so(5)$ and $sp_q(4) \equiv so_q(5)$ equation (27). Remark that in the last step of the embedding chain one can keep undeformed $SO(3)$ and the generators $l_\pm$ of the undeformed algebra of the angular momentum can be written, e.g., as

$$l_+ = \sum_k A_k^+(q, \{e_i^+, h_i\}) e_k^+ \tag{45}$$

where $A_k^+(q, \{e_i^+, h_i\})$ are operators taking values in $gl_q(6)$ (or $gl_q(5)$). By the way, let us recall that, from equation (17), as it is well known that the deformed algebras $su_q(n)$ and $sp_q(2n)$ can be written in terms of a bilinear of q-bosons, [21], it follows that one can write the following correspondences $su_q(n) \subset gl_q(n) \Leftrightarrow gl(n) \supset sl(n)$ and $sp_q(2n) \Leftrightarrow sp(2n)$, if each generator of algebras and q-algebras is written as a single bilinear of bosons and, respectively, of q-bosons. However, when a subalgebra is written in terms of the generators of the *mother algebra*, this property is lost. Let us point out that the problem in a deformation of the subalgebra of a subalgebra in an embedding chain of the type, e.g.,

$$gl_q(6) \supset L_q \supset so_q(3) \tag{46}$$

is that the generators $L_\pm$ cannot be written in terms of the generators of L_q, but only in terms of the generators of $gl_q(6)$. This peculiar feature is common to any deformed algebra when we consider an embedding chain with deformed singular subalgebra $L \subset G$. In conclusion, we have shown that it is possible to write the Cartan–Chevalley generators of $so_q(6)$ and $sl_q(3)$ in terms of the generators of $gl_q(6)$ and those of $so_q(5)$ in terms of $gl_q(5)$, but that it is not possible to extend further the procedure to obtain, in particular, $so_q(3)$, as was argued in [9, 13]. However, for the considered chains equation (8) we can go a step further writing the generators in terms of those of the *grandmother* $gl_q(6)$. Even if we have considered a particular case, so that our results are not quite general, the procedure used is general enough to be applied successfully to other physically relevant models, taking into account supersymmetric extensions. In the spirit of the use of spectrum generating algebra, one should write a Hamiltonian of a physical system as a sum of invariants of the q-algebras appearing in the embedding chain of the previous section. This can be done as the Casimir operators of $sl_q(n)$ [27, 28] and of $so_q(5)$ [29, 30] are known. For physical application one needs a self-adjoint Casimir. For q real this property is guaranteed in the realization we have used. In many applications of q-algebras, however, it seems that q being a phase is a favoured value (see [6]). For this value, due to the lack of invariance $q \leftrightarrow q^{-1}$ in our expressions, the product E^+E^- is no longer self-adjoint. In order to restore the self-adjointness of the Hamiltonian one has to sum the Casimir operators of $L_q(so_q(3))$ and of $L_{q^{-1}}(so_{q^{-1}}(3))$.[2] Let us briefly discuss some possible physical applications of the above proposed deformation scheme. As discussed in [6], it should be kept in mind that the q deformation parameter accounts more for deviations in the description of collective effects from the algebraic structure than for effects due to the 'deformation' of the nuclei. These deviations should be more present in heavy nuclei, with high angular excitations. We dispose presently of all the necessary mathematical tools, although the very complicated expressions of the generators and of the q-Casimir operators make the computations not at all easy and straightforward. Therefore, as a starting point, it might be useful to single out some specific nuclei, where the theoretical computations of the existing models (deformed or undeformed) show some discrepancy with the experimental

[2] The problems arising in the application of quantum algebra structure, which are never symmetric and seldom non-Hermitian, to describe composite physical systems have been discussed recently in [31]. The authors propose, in order to overcome the above difficulties, to symmetrize the quantum algebra operators. As a consequence the definition of tensor operators should no longer be given by equation (22). The proposed symmetric form satisfies the Hermiticity property for q real as well as imaginary.

data. In [14] for several of the considered Pt isotopes an inversion in the position of the energy levels 0_2^+ and 3_1^+ between the experimental and theoretical data appears. This may be a test for our deformation scheme. Another application could be the computation of energy spectra of ^{150}Sm, where, for high values of angular momentum, the theoretical fits of the IBM model and of the deformed version of this model proposed in [15] show some discrepancy with the data (see figure 4 in [15]).

As a final comment, if one considers transitions in the physical system induced by the q-tensor operator under $so_q(3)$ or, if the rotation group in physical space is left undeformed, by a tensor operator under $SO(3)$, one has to face the problem of the choice of the coproduct. In order to have the usual structure of q-tensor equation (22) or tensor operators, equation (22) in the limit $q \to 1$, the coproduct Δ has to be imposed in $so_q(3)$ or $SO(3)$ and cannot be inherited by that of G_q or L_q.

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